

## 4.7 - Cauchy-Euler Equations

**Definition:** A Cauchy-Euler equation is a differential equation of the form  $a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$ . Earlier in the chapter we found solutions of the form  $y = e^{mx}$ . Here we will find solutions of the form  $y = x^m$ .

Solve the given differential equation.

**Example:**  $x^2 y'' + 3xy' - 4y = 0$

$$y = x^m \Rightarrow y' = m x^{m-1}, y'' = m(m-1) x^{m-2}$$
$$x^2 m(m-1) x^{m-2} + 3x m x^{m-1} - 4 x^m = 0$$

$$m(m-1) x^m + 3m x^m - 4 x^m = 0$$

$$m(m-1) + 3m - 4 = 0$$

$$m^2 + 2m - 4 = 0$$

$$m^2 + 2m + 1 = 4 + 1 \Rightarrow (m+1)^2 = 5$$

$$m = -1 \pm \sqrt{5}$$

$$y = c_1 x^{-1+\sqrt{5}} + c_2 x^{-1-\sqrt{5}}$$

Generalizing: For  $ax^2 y'' + bx y' + cy = 0$ , the auxiliary equation is  $a m(m-1) + bm + c = 0$

Possible solutions:

Case I:  $m_1 \neq m_2 \in \mathbb{R}$

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case II:  $m_1 = m_2 = m$  (repeated)

$$y = c_1 x^m + c_2 x^m \ln x$$

← from reduction of order

Case III:  $m_1 = \alpha + \beta i$ ,  $m_2 = \alpha - \beta i$

$$y = x^\alpha (c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$$

Before,  $m_{1,2} = \alpha \pm \beta i$

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

Non homogeneous case:

$$ax^2 y'' + bxy' + cy = g(x)$$

std form

$$\hookrightarrow y'' + P(x)y' + Q(x)y = f(x)$$

↪ variation of parameters.

Example:  $4x^2y'' + y = 0$

$$\text{Let } y = x^m \Rightarrow y' = mX^{m-1}, y'' = m(m-1)x^{m-2}$$

$$4x^2 m(m-1)x^{m-2} + x^m = 0$$

$$x^m (4m^2 - 4m + 1) = 0$$

$$(2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}, \text{ repeated}$$

$$y_1 = x^{1/2}, y_2 = x^{1/2} \ln x$$

$$y = C_1 x^{1/2} + C_2 x^{1/2} \ln x$$

Example:  $x^2y'' - 7xy' + 41y = 0$

$$y = x^m, y' = mX^{m-1}, y'' = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-2} - 7x mX^{m-1} + 41x^m = 0$$

$$m^2 - m - 7m + 41 = 0 \Rightarrow m^2 - 8m + 41 = 0$$

$$(m-4)^2 = -41 + 16 \Rightarrow m-4 = \pm 5i$$

$$m = 4 \pm 5i$$

$$y = x^4 [C_1 \cos(5 \ln x) + C_2 \sin(5 \ln x)]$$

A Cauchy-Euler equation can be converted to an equation with constant coefficients using the substitution  $x = e^t$  and the Chain Rule.

Rationale: The general solution to

$ay'' + by' + cy = 0$ ,  $a, b, c$  constant is

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$  (\*). The general

solution to  $ax^2 y'' + bxy' + cy = 0$  ( $a, b, c$  constant)

is  $y = C_1 x^{m_1} + C_2 x^{m_2}$ . But  $x = e^{\ln x}$   
 $\Rightarrow x^{m_1} = e^{m_1 \ln x} \Rightarrow y = C_1 e^{m_1 \ln x} + C_2 e^{m_2 \ln x}$

If we let  $x = e^t$ , we get

$y = C_1 e^{m_1 t} + C_2 e^{m_2 t}$ , resembling (\*).

Example: Use the substitution  $x = e^t$  to transform the given Cauchy-Euler equation to a differential equation with constant coefficients. Solve the original equation by solving the new equation using techniques previously learned.

$$x^2 y'' - 9xy' + 25y = 0 \quad y = x^m; \quad x = e^t \Rightarrow t = \ln x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} \quad \text{with } t(x) = \ln x$$

$y(x)$  becomes  $y(t(x))$

$$\frac{d^2 y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d}{dx} \left( \frac{dy}{dt} \right)$$

$$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \left( \frac{d^2y}{dt^2} \frac{1}{x} \right) = \frac{1}{x^2} \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right)$$
$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 9 \frac{dy}{dt} + 25y = 0$$

$$\frac{d^2y}{dt^2} - 10 \frac{dy}{dt} + 25y = 0$$

$$(m-5)^2 = 0 \Rightarrow m=5, \text{ repeated}$$

$$y = c_1 e^{5t} + c_2 t e^{5t}$$

$$y = c_1 x^5 + c_2 \ln x x^5$$

Alternatively:  $x^2 y'' - 9xy' + 25y = 0$   $y = x^m$

$$m(m-1) - 9m + 25 = 0$$

$$m^2 - 10m + 25 = 0 \Rightarrow (m-5)^2 = 0$$